Weighted Hardy and singular operators in Morrey spaces

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Abstract

We study the weighted boundedness of the Cauchy singular integral operator S_Γ in Morrey spaces $L^{p,\lambda}(\Gamma)$ on curves satisfying the arc-chord condition, for a class of "radial type" almost monotonic weights. The non-weighted boundedness is shown to hold on an arbitrary Carleson curve. We show that the weighted boundedness is reduced to the boundedness of weighted Hardy operators in Morrey spaces $L^{p,\lambda}(0,\ell), \ell > 0$. We find conditions for weighted Hardy operators to be bounded in Morrey spaces. To cover the case of curves we also extend the boundedness of the Hardy-Littlewood maximal operator in Morrey spaces, known in the Euclidean setting, to the case of Carleson curves.

1 Introduction

The well known Morrey spaces $\mathcal{L}^{p,\lambda}$ introduced in [30] in relation to the study of partial differential equations, and presented in various books, see [18], [26], [47], were widely investigated during last decades, including the study of classical operators of harmonic analysis - maximal, singular and potential operators - in these spaces; we refer for instance to the papers [1], [2], [3], [5], [9], [12], [13], [32], [33], [35], [36], [37], [38], [40], [43], [44], [45], where Morrey spaces on metric measure spaces may be also found. In particular, for the boundedness of the maximal operator in Morrey spaces we refer to [9], while the boundedness of Calderon-Zygmund type singular operators is known from [37], [38], [44].

Meanwhile weighted estimations of these operators in Morrey spaces were not studied (to our surprise, we did not find any such weighted result for maximal and singular operators in the literature). We are mainly interested in weighted estimations of singular operators. In this paper we deal with the one-dimensional case and study the weighted boundedness of the Cauchy singular integral operator

$$S_\Gamma f(t) = \frac{1}{\pi} \int_\Gamma \frac{f(\tau)\, d\tau}{\tau - t} \tag{1.1}$$

along curves on complex plane, such a weighted estimation being a key point for applications to the solvability theory of singular integral equations. We refer to [15], [31], [19], [20], [7] for this theory.

We obtain weighted estimates of the operator S_Γ in Morrey spaces $\mathcal{L}^{p,\lambda}(\Gamma)$ along an arbitrary curve satisfying the arc-chord condition, in case of weights ϱ of the form

$$\varrho(t) = \prod_{k=1}^{N} \varphi_k(|t - t_k|), \quad t_k \in \Gamma, \tag{1.2}$$

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with almost monotone functions φ_k, the necessary and sufficient condition for the boundedness being given in terms of the Matuszewska-Orlicz indices of these functions.

We start with the case where $\Gamma = [0, \ell]$ is an interval of the real axis. We make use of the known non-weighted boundedness of S_Γ in this case, which enables us to reduce the boundedness of S_Γ with weight (1.2) to the boundedness of weighted Hardy operators. We prove their boundedness in the Morrey space $\mathcal{L}^{p,\lambda}(0, \ell)$. Surprisingly we did not find any statement on the boundedness of Hardy operators in Morrey spaces in the literature. In the context of Morrey spaces, Hardy operators seem to have appeared only in [8] in a different aspect: the problem of the boundedness of the maximal operator in local Morrey spaces was reduced to an L_p-boundedness of Hardy operators on a cone of monotone functions.

To cover the case of an arbitrary curve Γ we have to prove first the non-weighted boundedness of the singular operator on curves, which was unknown, up to our knowledge. To this end, we first prove the boundedness of the maximal operator along a Carleson curve, in $\mathcal{L}^{p,\lambda}(\Gamma)$, but give the proof within the frameworks of homogeneous spaces with constant dimension, Carleson curves being examples of such a space. Then we derive the non-weighted boundedness of S_Γ via the Alvarez-Pérez-type pointwise estimate

$$M^{\#}(|S_\Gamma f|^s)(t) \leq C[Mf(t)]^s, \quad 0 < s < 1. \tag{1.3}$$

known in the Euclidean setting for Calderon-Zygmund singular operators ([4]) and extended to the case of the operator S_Γ along Carleson curves in [24], Proposition 6.2.

The paper is organized as follows. In Section 2 we provide necessary preliminaries on Morrey spaces on metric measure spaces, on almost monotonic weights and their Matuszewska-Orlicz indices. In Section 3 we give sufficient conditions of the boundedness of weighted Hardy operators in Morrey spaces $\mathcal{L}^{p,\lambda}(0, \ell)$ in terms of the indices of the weight. These conditions are necessary in the case of power weights. In Section 4 we give sufficient conditions of weighted boundedness of the singular operator along $(0, \ell)$, which prove to be also necessary for power weights. In Section 5 we first extend Chiarenza-Frasca's proof ([9]) of the boundedness of the maximal operator to the case of metric measure spaces X with constant dimension and prove the Fefferman-Stein inequality $\|Mf\|_{L^{p,\lambda}(X)} \leq C\|M^{\#}f\|_{L^{p,\lambda}(X)}$, to derive the non-weighted boundedness of S_Γ via (1.3). Finally in Section 6, we prove the weighted boundedness of S_Γ.

2 Preliminaries

2.1 Morrey spaces on homogeneous spaces

Let (X, d, μ) be a homogeneous metric measure space with quasidistance d and measure μ. We refer to [11], [17], [21] for analysis in homogeneous spaces. Morrey spaces on metric measure spaces were studied in [5], [34], [48]. Our main goal in the sequel is the case where X is a Carleson curve in the complex plane with μ an arc-length, although some auxiliary statements will be given in a more general setting. By this reason we restrict ourselves to the case where X has constant dimension: there exists a number $N > 0$ (not necessarily integer) such that

$$C_1 r^N \leq \mu B(x, r) \leq C_2 r^N, \tag{2.1}$$

where the constants $C_1 > 0$ and $C_2 > 0$ do not depend on $x \in X$ and $r > 0$. In this case the Morrey space $\mathcal{L}^{p,\lambda}(X)$ may be defined by the norm:

$$\|f\|_{p,\lambda} = \sup_{x \in X, r > 0} \left\{ \frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p \, d\mu(y) \right\}^{\frac{1}{p}}, \tag{2.2}$$

where $1 \leq p < \infty$ and $0 \leq \lambda < N$ and the standard notation $B(x,r) = \{y \in X : d(x,y) : r\}$ is used.

2.2 The case of Carleson curves

Let Γ be a bounded rectifiable curve on the complex plane $\mathbb{C}$. We denote $\tau = t(\sigma), \quad t = t(s)$ where σ and s stand for the arc abscissas of the points τ and t, and $d\mu(\tau) = d\sigma$ will stand for the arc-measure on Γ. We also introduce the notation

$$\Gamma(t,r) = \{\tau \in \Gamma : |\tau - t| < r\} \quad \text{and} \quad \Gamma_*(t,r) = \{\tau \in \Gamma : |\sigma - s| < r\},$$

so that $\Gamma_*(t,r) \subseteq \Gamma(t,r)$, and denote $\ell = \mu\Gamma = $ lengths of Γ.

Definition 2.1. A a curve Γ is said to be a *Carleson curve*, if

$$\mu\Gamma(t,r) \leq Cr$$

for all $t \in \Gamma$ and $r > 0$, where $C > 0$ does not depend on t and r. A curve Γ is said to satisfy *the arc-chord condition at a point $t_0 = t(s_0) \in \Gamma$*, if there exists a constant $k > 0$, not depending on t such that

$$|s - s_0| \leq k|t - t_0|, \quad t = t(s) \in \Gamma. \tag{2.3}$$

Finally, a curve Γ is said to satisfy the *(uniform) arc-chord condition*, if

$$|s - \sigma| \leq k|t - \tau|, \quad t = t(s), \tau = t(\sigma) \in \Gamma. \tag{2.4}$$

In the sequel Γ is always assumed to be a Carleson curve.

The Morrey spaces $\mathcal{L}^{p,\lambda}(\Gamma)$ on Γ are defined in the usual way, as in (2.2), via the norm

$$\|f\|_{p,\lambda} = \sup_{t \in \Gamma, r > 0} \left\{ \frac{1}{r^\lambda} \int_{\Gamma(t,r)} |f(\tau)|^p \, d\mu(\tau) \right\}^{\frac{1}{p}}, \tag{2.5}$$

where $1 \leq p < \infty$ and $0 \leq \lambda \leq 1$. For brevity we denote $\|f\|_p = \left(\int_\Gamma |f(\tau)|^p d\mu(\tau) \right)^{\frac{1}{p}}$, so that

$$\|f\|_{p,\lambda} = \sup_{t,r} \left\| \frac{\chi_{\Gamma(t,r)}(\cdot)}{r^{\frac{\lambda}{p}}} f(\cdot) \right\|_p. \tag{2.6}$$

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Remark 2.2. One may define another version $\mathcal{L}_*^{p,\lambda}(\Gamma)$ of the Morrey space, in terms of the arc neigbourhood $\Gamma_*(t,r)$ of the point $t \in \Gamma$, by the norm

$$\|f\|_{p,\lambda}^* = \sup_{t\in\Gamma, r>0} \left\{ \frac{1}{r^\lambda} \int\limits_{\Gamma_*(t,r)} |f(\tau)|^p \, d\mu(\tau) \right\}^{\frac{1}{p}} \tag{2.7}$$

so that $\mathcal{L}_*^{p,\lambda}(\Gamma) \subseteq \mathcal{L}^{p,\lambda}(\Gamma)$ in case of an arbitrary curve. These spaces coincide, up to equivalence of the norms, when Γ satisfies the arc-chord condition. If Γ has casps, these spaces may be different. If, for instance, a bounded curve Γ satisfies the condition

$$C|s - \sigma|^a \leq |t - \tau|, \quad C > 0$$

for some $a \geq 1$, then $\mathcal{L}^{p,a\lambda}(\Gamma) \subseteq \mathcal{L}_*^{p,\lambda}(\Gamma) \subseteq \mathcal{L}^{p,\lambda}(\Gamma)$.

Lemma 2.3. *Let Γ be bounded rectifiable curve. For the power function $|t - t_0|^\gamma$, $t_0 \in \Gamma$, to belong to the Morrey space $\mathcal{L}^{p,\lambda}(\Gamma)$, $1 \leq p < \infty$, $0 < \lambda < 1$, the condition*

$$\gamma \geq \frac{\lambda - 1}{p} \tag{2.8}$$

is necessary. It is also sufficient if Γ is a Carleson curve

Proof. *The necessity part.* Let $|t - t_0|^\gamma \in \mathcal{L}^{p,\lambda}(\Gamma)$. We suppose that $\gamma < 0$, since there is nothing to prove when $\gamma \geq 0$. With $t_0 = t(s_0)$ we have

$$\||\tau - t_0|^\gamma\| \geq \sup_{r>0} \left(\frac{1}{r^\lambda} \int\limits_{\Gamma(t_0,r)} |\tau - t_0|^{\gamma p} d\mu(\tau) \right)^{\frac{1}{p}} \geq \sup_{r>0} \left(\frac{1}{r^\lambda} \int\limits_{|\sigma - s_0| < r} |\sigma - s_0|^{\gamma p} d\sigma \right)^{\frac{1}{p}}, \tag{2.9}$$

where we have taken into account that $\Gamma(t,r) \supseteq \Gamma_*(t,r)$ and γ is negative. Since $|t - t_0|^\gamma \in L^p(\Gamma)$, by similar arguments we see that $\gamma p > -1$. Then we get

$$\||\tau - t_0|^\gamma\|_{p,\lambda} \geq \left(\frac{2}{\gamma p + 1} \right)^{\frac{1}{p}} \sup_{r>0} r^{\frac{\gamma p + 1 - \lambda}{p}}$$

which may be finite only when $\gamma p + 1 - \lambda \geq 0$.

The sufficiency part. Let $\gamma \geq \frac{\lambda-1}{p}$. Again we may assume that γ is negative. To estimate

$$\||\tau - t_0|^\gamma\|_{p,\lambda} = \sup_{t,r} \left(\frac{1}{r^\lambda} \int\limits_{\Gamma(t,r)} |\tau - t_0|^{\gamma p} d\mu(\tau) \right)^{\frac{1}{p}},$$

we distinguish the cases $|t - t_0| > 2r$ and $|t - t_0| \leq 2r$. In the first case we have $|\tau - t_0| \geq |t - t_0| - |\tau - t| > r$ so that $|\tau - t_0|^{\gamma p} < r^\gamma$ and then

$$\||\tau - t_0|^\gamma\|_{p,\lambda} \leq \sup_{t,r} \left(r^{\gamma p - \lambda} \int\limits_{\Gamma(t,r)} d\mu(\tau) \right)^{\frac{1}{p}} = Cr^{\frac{\gamma p - \lambda - 1}{p}} < \infty$$

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where we used the fact that Γ is a Carleson curve. In the case $|t - t_0| \leq 2r$ we have $\Gamma(t,r) \subseteq \Gamma(t_0, 3r)$. Then

$$\||\tau - t_0|^\gamma\|_{p,\lambda} \leq \sup_r \left(\frac{1}{r^\lambda} \sum_{k=0}^\infty \int_{\Gamma_k(t_0,r)} |\tau - t_0|^{\gamma p} d\mu(\tau) \right)^{\frac{1}{p}},$$

where $\Gamma_k(t_0, r) = \{\tau : 3 \cdot 2^{-k-1}r < |\tau - t_0| < 3 \cdot 2^{-k}r\}$. Hence

$$\||\tau - t_0|^\gamma\|_{p,\lambda} \leq C \sup_r \left(\frac{1}{r^{\lambda - \gamma p}} \sum_{k=0}^\infty \frac{1}{2^{\gamma p k}} \int_{\Gamma(t_0, 2^{-k+1}r)} d\mu(\tau) \right)^{\frac{1}{p}}$$

and we arrive at the conclusion by standard arguments. $\qquad\square$

Remark 2.4. The case $\lambda > 0$ differs from the case $\lambda = 0$: when $\lambda = 0$, condition (2.8) must be replaced by the condition $\gamma > -\frac{1}{p}$.

In the limiting case $\mu = \frac{\lambda - 1}{p}$ admitted in Lemma 2.3 for power functions, it is not possible to take a power-logarithmic function, as shown in the next lemma.

Lemma 2.5. *Let Γ be bounded rectifiable curve. The function $|t - t_0|^{\frac{\lambda-1}{p}} \ln^\nu \frac{A}{|t-t_0|}$, where $t_0 \in \Gamma, \nu > 0$ and $A \geq D$, does not belong to $\mathcal{L}^{p,\lambda}(\Gamma)$.*

Proof. As in (2.9), we have

$$\left\| |\tau - t_0|^{\frac{\lambda-1}{p}} \ln^\nu \frac{A}{|\tau - t_0|} \right\|_{p,\lambda} \geq \sup_{r>0} \left(\frac{1}{r^\lambda} \int_{|\tau - t_0|<r} |\tau - t_0|^{\lambda-1} \ln^{\nu p} \frac{A}{|\tau - t_0|} d\mu(\tau) \right)^{\frac{1}{p}} \qquad (2.10)$$

$$\geq \sup_{r>0} \left(\frac{1}{r^\lambda} \int_{|\sigma - s_0|<r} |\sigma - s_0|^{\lambda-1} \ln^{\nu p} \frac{A}{|\sigma - s_0|} d\sigma \right)^{\frac{1}{p}}$$

$$= \sup_{r>0} \left(2 \int_0^1 t^{\lambda-1} \left(\ln \frac{A}{r} + \ln \frac{1}{t} \right)^{\nu p} dt \right)^{\frac{1}{p}} \geq \sup_{0<r<\delta} \left(2 \ln^{\nu p} \frac{A}{r} \int_0^1 t^{\lambda-1} dt \right)^{\frac{1}{p}} = \infty.$$

$\qquad\square$

Remark 2.6. Statements similar to Lemmas 2.3 and 2.5 hold also for Morrey spaces over bounded sets Ω in $\mathbb{R}^n$:

1. *The power function $|x - x_0|^\gamma$, where $x_0 \in \Omega$ belongs to the Morrey space $\mathcal{L}^{p,\lambda}(\Omega), 1 \leq p < \infty, 0 < \lambda < n$, if and only if $\gamma \geq \frac{\lambda - n}{p}$. In the case $x_0 \in \partial\Omega$, the condition $\gamma \geq \frac{\lambda - n}{p}$ remains sufficient; it is also necessary if the point x_0 is a regular point of the boundary in the sense that $|\{y \in \Omega : |y - x_0| < r\}| \sim cr^n$.*

2. *The function $|x - x_0|^{\frac{\lambda - n}{p}} \ln^\nu \frac{D}{|x-x_0|}, D > diam \, \Omega$, where $x_0 \in \Omega$ or x_0 is a regular point of $\partial\Omega$, does not belong to $\mathcal{L}^{p,\lambda}(\Omega)$.*

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2.3   On admissible weight functions

In the sequel, when studying the singular operator S_Γ along a curve Γ in weighted Morrey space, we will deal with weights of the form

$$\varrho(t) = \prod_{k=1}^{N} \varphi_k(|t - t_k|), \quad t_k \in \Gamma. \tag{2.11}$$

We introduce below the class of weight functions $\varphi_k(x)$, $x \in [0, \ell]$, admitted for our goals.

Although the functions φ_k should be defined only on $[0, d]$, where $d = \mathrm{diam}\Gamma = \sup_{t,\tau \in \Gamma} |t - \tau| < \ell$, everywhere below we consider them as defined on $[0, \ell]$.

Definition 2.7.
1) By W we denote the class of continuous and positive functions $\varphi(x)$ on $(0, \ell]$,
2) by W_0 we denote the class of functions $\varphi \in W$ such that $\lim_{x \to 0} \varphi(x) = 0$ and $\varphi(x)$ is almost increasing;
3) by $\widetilde{W}$ we denote the class of functions $\varphi \in W$ such $x^\alpha \varphi(x) \in W_0$ for some $\alpha = \alpha(\varphi) > 0$.

Definition 2.8. Let $x, y \in (0, \ell]$ and $x_+ = \max(x, y)$, $x_- = \min(x, y)$. By $\mathbf{V}_{\pm\pm}$ we denote the classes of functions $\varphi \in W$ defined by the following conditions

$$\mathbf{V}_{++}: \qquad \left| \frac{\varphi(x) - \varphi(y)}{x - y} \right| \le C \frac{\varphi(x_+)}{x_+}, \tag{2.12}$$

$$\mathbf{V}_{--}: \qquad \left| \frac{\varphi(x) - \varphi(y)}{x - y} \right| \le C \frac{\varphi(x_-)}{x_-}, \tag{2.13}$$

$$\mathbf{V}_{+-}: \qquad \left| \frac{\varphi(x) - \varphi(y)}{x - y} \right| \le C \frac{\varphi(x_+)}{x_-}, \tag{2.14}$$

$$\mathbf{V}_{-+}: \qquad \left| \frac{\varphi(x) - \varphi(y)}{x - y} \right| \le C \frac{\varphi(x_-)}{x_+}. \tag{2.15}$$

Obviously, $\mathbf{V}_{++} \subset \mathbf{V}_{+-}$ and $\mathbf{V}_{-+} \subset \mathbf{V}_{--}$.

Let $0 < y < x \le \ell$. It is easy to check that in the case of power function $\varphi(x) = x^\alpha, \alpha \in \mathbb{R}^1$, we have

$$|x^\alpha - y^\alpha| \le C(x - y)x^{\alpha - 1} \iff \alpha \ge 0 \tag{2.16}$$

$$|x^\alpha - y^\alpha| \le C(x - y)y^{\alpha - 1} \iff \alpha \le 1, \tag{2.17}$$

$$|x^\alpha - y^\alpha| \le C(x - y)\frac{x^\alpha}{y} \iff \alpha \ge -1 \tag{2.18}$$

$$|x^\alpha - y^\alpha| \le C(x - y)\frac{y^\alpha}{x} \iff \alpha \le 0, \tag{2.19}$$

where the constant $C > 0$ does not depend on x and y. Thus,

$$x^\alpha \in \mathbf{V}_{++} \iff \alpha \ge 0, \qquad x^\alpha \in \mathbf{V}_{--} \iff \alpha \le 1$$

and
$$x^\alpha \in \mathbf{V}_{+-} \iff \alpha \geq -1, \qquad x^\alpha \in \mathbf{V}_{-+} \iff \qquad \alpha \leq 0.$$

In the sequel we will mainly work with the classes $\mathbf{V}_{++}$ and $\mathbf{V}_{-+}$.

We also denote
$$W_1 = \left\{ \varphi \in W : \frac{\varphi(x)}{x} \text{ is almost decreasing} \right\}.$$

Remark 2.9. Note that functions $\varphi \in W_1$ satisfy the doubling condition $\varphi(2x) \leq C\varphi(x)$. For a function $\varphi \in W_1$, condition (2.12) yields condition (2.13), that is $W_1 \cap \mathbf{V}_{++} \subseteq W_1 \cap \mathbf{V}_{--}$. The inverse embedding may be not true, as the above example of the power functions in (2.16) -(2.19) shows.

In the following lemma we show that conditions (2.12) and (2.13) are fulfilled automatically not only for power functions, but for an essentially larger class of functions (which in particular may oscillate between two power functions with different exponents). Note that the information about this class is given in terms of increasing or decreasing functions, without the word "almost". Two statements *i)* and *ii)* in Lemma 2.10 reflect in a sense the modelling cases (2.16) and (2.17).

Lemma 2.10. *Let $\varphi \in W$. Then*
i) $\varphi \in V_{++}$ *in the case φ is increasing and the function $\frac{\varphi(x)}{x^\nu}$ is decreasing for some $\nu \geq 0$;*
ii) $\varphi \in V_{--}$ *in the case $\frac{\varphi(x)}{x}$ is decreasing and there exist a number $\mu \geq 0$ such that $x^\mu \varphi(x)$ is increasing;*
iii) $\varphi \in V_{-+}$ *in the case $\varphi(x)$ is decreasing and there exist a number $\mu \geq 0$ such that $x^\mu \varphi(x)$ is increasing.*

Proof. The case *i)*. Let $0 < y < x \leq \ell$. Since $\frac{\varphi(x)}{x^\nu}$ is decreasing, we have $1 - \frac{\varphi(y)}{\varphi(x)} \leq 1 - \frac{y^\nu}{x^\nu}$ or
$$\varphi(x) - \varphi(y) \leq \varphi(x)\frac{x^\nu - y^\nu}{x^\nu} \leq C\varphi(x)\frac{x-y}{x}$$

by (2.16). Since $\varphi(x) - \varphi(y) \geq 0$ in the case $\mu = 0$, we arrive at (2.12).

The cases *ii)* and *iii)*. Taking again $y < x$, by the case *i)* we have that (2.12) holds for the function $x^\mu \varphi(x)$, that is,
$$\left| \varphi(x) - \frac{y^\mu}{x^\mu}\varphi(y) \right| \leq C\frac{\varphi(x)}{x}(x-y).$$

Then
$$|\varphi(x) - \varphi(y)| \leq C\frac{\varphi(x)}{x}(x-y) + \frac{x^\mu - y^\mu}{x^\mu}\varphi(y)$$
$$\leq C\frac{\varphi(x)}{x}(x-y) + C\frac{x-y}{x}\varphi(y)$$

by (2.16). In the case *ii)* we use the fact that $\frac{\varphi(x)}{x} \leq \frac{\varphi(y)}{y}$ and arrive at (2.13). In the case *iii)* we have $\varphi(x) \leq \varphi(y)$ and arrive at (2.15). $\square$

In the case of differentiable functions $\varphi(x), x > 0$, we arrive at the following sufficient conditions for φ to belong to the classes $\mathbf{V}_{++}, \mathbf{V}_{-+}$.

Lemma 2.11. *Let* $\varphi \in W \cap C^1((0, \ell])$. *If there exist* $\varepsilon > 0$ *and* $\nu \geq 0$ *such that*

$$0 \leq \frac{\varphi'(x)}{\varphi(x)} \leq \frac{\nu}{x} \quad \text{for} \quad 0 < x \leq \varepsilon,$$

then $\varphi \in \mathbf{V}_{++}$. *If there exist* $\varepsilon > 0$ *and* $\mu \geq 0$ *such that*

$$-\frac{\mu}{x} \leq \frac{\varphi'(x)}{\varphi(x)} \leq 0 \quad \text{for} \quad 0 < x \leq \varepsilon,$$

then $\varphi \in \mathbf{V}_{++}$.

Proof. Since $\min_{x \in [\varepsilon, \ell]} \varphi(x) > 0$, by the differentiability of $\varphi(x)$ beyond the origin, inequalities (2.12), (2.13) hold automatically when $x \geq \varepsilon$, so the conditions of Lemma 2.10 should be checked only on $(0, \varepsilon]$. It suffices to note that $\frac{\varphi'(x)}{\varphi(x)} \leq \frac{\nu}{x} \iff \left[\frac{\varphi(x)}{x^\nu}\right]' \leq 0$ and $\frac{\varphi'(x)}{\varphi(x)} \geq -\frac{\mu}{x} \iff [x^\mu \varphi(x)]' \geq 0$. $\qquad \square$

Example 2.12. Let $\alpha, \beta \in \mathbb{R}^1$ and $A > \ell$. Then

$$x^\alpha \left(\ln \frac{A}{x}\right)^\beta \in \begin{cases} \mathbf{V}_{++}, & \text{if } \alpha > 0, \beta \in \mathbb{R}^1 \text{ or } \alpha = 0 \text{ and } \beta \leq 0 \\ \mathbf{V}_{-+}, & \text{if } \alpha < 0, \beta \in \mathbb{R}^1 \text{ or } \alpha = 0 \text{ and } \beta \geq 0. \end{cases}$$

2.4 Matuszewska-Orlicz type indices

It is known that the property of a function to be almost increasing or almost decreasing after the multiplication (division) by a power function is closely related to the notion of the so called Matuszewska-Orlicz indices. We refer to [22], [25], [27] (p.20), [28], [29], [41], [42] for the properties of the indices of such a type. For a function $\varphi \in W_0$, the numbers

$$m(\varphi) = \sup_{0 < x < 1} \frac{\ln \left(\limsup_{h \to 0} \frac{\varphi(hx)}{\varphi(h)}\right)}{\ln x} = \lim_{x \to 0} \frac{\ln \left(\limsup_{h \to 0} \frac{\varphi(hx)}{\varphi(h)}\right)}{\ln x}$$

and

$$M(\varphi) = \sup_{x > 1} \frac{\ln \left(\limsup_{h \to 0} \frac{\varphi(hx)}{\varphi(h)}\right)}{\ln x} = \lim_{x \to \infty} \frac{\ln \left(\limsup_{h \to 0} \frac{\varphi(hx)}{\varphi(h)}\right)}{\ln x}$$

are known as *the Matuszewska-Orlicz type lower and upper indices* of the function $\varphi(r)$. Note that in this definition $\varphi(x)$ need not to be an N-function: only its behaviour at the origin is of importance. Observe that

$$0 \leq m(\varphi) \leq M(\varphi) \leq \infty \text{ for } \varphi \in W_0.$$

It is obvious that

$$m[x^\lambda \varphi(x)] = \lambda + m(\varphi), \quad M[x^\lambda \varphi(x)] = \lambda + M(\varphi), \qquad \lambda \in \mathbb{R}^1. \tag{2.20}$$

Consequently,

$$-\infty < m(\varphi) \le M(\varphi) \le \infty \quad \text{for} \quad \varphi \in \widetilde{W}.$$

Definition 2.13. We say that a function $\varphi \in W_0$ belongs to the Zygmund class $\mathbb{Z}^\beta$, $\beta \in \mathbb{R}^1$, if

$$\int_0^h \frac{\varphi(x)}{x^{1+\beta}} dx \le c \frac{\varphi(h)}{h^\beta},$$

and to the Zygmund class $\mathbb{Z}_\gamma$, $\gamma \in \mathbb{R}^1$, if

$$\int_h^\ell \frac{\varphi(x)}{x^{1+\gamma}} dx \le c \frac{\varphi(h)}{h^\gamma}.$$

We also denote

$$\mathbb{Z}_\gamma^\beta := \mathbb{Z}^\beta \bigcap \mathbb{Z}_\gamma,$$

the latter class being also known as Bary-Stechkin-Zygmund class [6].

The following statement is known, see [22], Theorems 3.1 and 3.2.

Theorem 2.14. *Let $\varphi \in \widetilde{W}$ and $\beta, \gamma \in \mathbb{R}^1$. Then*

$$\varphi \in \mathbb{Z}^\beta \Longleftrightarrow m(\varphi) > \beta \quad \text{and} \quad \varphi \in \mathbb{Z}_\gamma \Longleftrightarrow M(\varphi) < \gamma.$$

Besides this

$$m(\varphi) = \sup\left\{ \delta > 0 : \ \frac{\varphi(x)}{x^\delta} \quad \text{is almost increasing} \right\}, \tag{2.21}$$

and

$$M(\varphi) = \inf\left\{ \lambda > 0 : \ \frac{\varphi(x)}{x^\lambda} \quad \text{is almost decreasing} \right\}. \tag{2.22}$$

Remark 2.15. Theorem 2.14 was formulated in [22] for $\beta \ge 0, \gamma > 0$ and $\varphi \in W_0$. It is evidently true when these exponents are negative, and for $\varphi \in \widetilde{W}$, the latter being obvious by the definition of the class $\widetilde{W}$ and formulas (2.20).

3 Weighted Hardy operators in Morrey spaces

3.1 The case of power weights

Let

$$H_\beta f(x) = x^{\beta-1} \int_0^x \frac{f(t)dt}{t^\beta}, \quad \mathcal{H}_\beta f(x) = x^\beta \int_x^\ell \frac{f(t)dt}{t^{\beta+1}}. \tag{3.1}$$

Theorem 3.1. *Let $0 < \ell \le \infty$. The operators H_β and $\mathcal{H}_\beta$ are bounded in the Morrey space $\mathcal{L}^{p,\lambda}([0,\ell])$, $1 \le p < \infty, 0 \le \lambda < 1$, if and only if*

$$\beta < \frac{\lambda}{p} + \frac{1}{p'} \quad \text{and} \quad \beta > \frac{\lambda}{p} - \frac{1}{p}, \tag{3.2}$$

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respectively.

Proof.
"If" part. We may assume that $f(x) \geq 0$. First we observe that

$$H_\beta f(x) = \int\limits_0^1 \frac{f(xt)dt}{t^\beta} \quad \text{and} \quad \mathcal{H}_\beta f(x) = \int\limits_1^{\frac{\ell}{x}} \frac{f(xt)dt}{t^{\beta+1}} \leq \int\limits_1^\infty \frac{f(xt)dt}{t^{\beta+1}}$$

under the assumption that $f(x)$ is continued as $f(x) \equiv 0$ for $x > \ell$ in the inequality for $\mathcal{H}_\beta f(x)$. For $H_\beta f$ we have

$$\|H_\beta f\|_{p,\lambda} = \sup_{x,r} \left\| \frac{\chi_{B(x,r)}(y)}{r^\lambda} H_\beta f(y) \right\|_p = \sup_{x,r} \left\{ \frac{1}{r^\lambda} \int\limits_0^\infty \left| \chi_{B(x,r)}(y) \int\limits_0^1 \frac{f(yt)}{t^\beta} dt \right|^p dy \right\}^{\frac{1}{p}}.$$

Then by Minkowsky inequality we obtain

$$\|H_\beta f\|_{p,\lambda} \leq \sup_{x,r} \int\limits_0^1 \frac{dt}{t^\beta} \left\{ \int\limits_0^\infty \left| \frac{\chi_{B(x,r)}(y)}{r^\lambda} f(yt) \right|^p dy \right\}^{\frac{1}{p}}.$$

Hence, by the change of variables we get

$$\|H_\beta f\|_{p,\lambda} \leq \sup_{x,r} \int\limits_0^1 \frac{dt}{t^{\beta+\frac{1}{p}}} \left\{ \int\limits_0^\infty \left| \frac{\chi_{B(x,r)}\left(\frac{y}{t}\right)}{r^{\frac{\lambda}{p}}} f(y) \right|^p dy \right\}^{\frac{1}{p}}.$$

It is easy to see that

$$\chi_{B(x,r)}\left(\frac{y}{t}\right) = \chi_{B(tx,tr)}(y).$$

Therefore,

$$\|H_\beta f\|_{p,\lambda} \leq \sup_{x,r} \int\limits_0^1 \frac{dt}{t^{\beta+\frac{1}{p}}} \left\{ \int\limits_0^\infty \left| \frac{\chi_{B(tx,tr)}(y)}{r^{\frac{\lambda}{p}}} f(y) \right|^p dy \right\}^{\frac{1}{p}}$$

$$= \sup_{x,r} \int\limits_0^1 \frac{dt}{t^{\beta+\frac{1-\lambda}{p}}} \left\{ \int\limits_0^\infty \left| \frac{\chi_{B(tx,tr)}(y)}{(tr)^{\frac{\lambda}{p}}} f(y) \right|^p dy \right\}^{\frac{1}{p}}$$

$$\leq \int\limits_0^1 \frac{dt}{t^{\beta+\frac{1-\lambda}{p}}} \sup_{x,r} \left\{ \int\limits_0^\infty \left| \frac{\chi_{B(x,r)}(y)}{r^\lambda} f(y) \right|^p dy \right\}^{\frac{1}{p}} = \frac{1}{\frac{\lambda}{p} + \frac{1}{p'} - \beta} \|f\|_{p,\lambda}$$

Similarly for the operator $\mathcal{H}_\beta$ we obtain

$$\|\mathcal{H}_\beta f\|_{p,\lambda} = \sup_{x,r} \left\| \frac{\chi_{B(x,r)}(y)}{r^{\frac{\lambda}{p}}} \mathcal{H}_\beta f(y) \right\|_p \leq \sup_{x,r} \int\limits_1^\infty \frac{dt}{t^{1+\beta}} \left\{ \int\limits_0^\infty \left| \frac{\chi_{B(x,r)}(y)}{r^{\frac{\lambda}{p}}} f(yt) \right|^p dy \right\}^{\frac{1}{p}}$$

$$= \sup_{x,r} \int_1^\infty \frac{dt}{t^{1+\beta+\frac{1}{p}}} \left\{ \int_0^\infty \left| \frac{\chi_{B(x,r)}\left(\frac{y}{t}\right)}{r^{\frac{\lambda}{p}}} f(y) \right|^p dy \right\}^{\frac{1}{p}}$$

$$= \sup_{x,r} \int_1^\infty \frac{dt}{t^{1+\beta+\frac{1}{p}}} \left\{ \int_1^\infty \left| \frac{\chi_{B(tx,tr)}(y)}{r^{\frac{\lambda}{p}}} f(y) \right|^p dy \right\}^{\frac{1}{p}} = \sup_{x,r} \int_1^\infty \frac{dt}{t^{1+\beta+\frac{1}{p}-\frac{\lambda}{p}}} \left\{ \int_0^\infty \left| \frac{\chi_{B(tx,tr)}(y)}{(tr)^{\frac{\lambda}{p}}} f(y) \right|^p dy \right\}^{\frac{1}{p}}$$

$$\leq \int_1^\infty \frac{dt}{t^{1+\beta+\frac{1-\lambda}{p}}} \sup_{x,r} \left\{ \int_0^\infty \left| \frac{\chi_{B(x,r)}(y)}{r^{\frac{\lambda}{p}}} f(y) \right|^p dy \right\}^{\frac{1}{p}} = \frac{1}{\beta+\frac{1-\lambda}{p}} \|f\|_{p,\lambda}$$

"Only if" part. The necessity of the condition $\beta < \frac{\lambda}{p} + \frac{1}{p'}$ for the operator H_β is well known in the case $\lambda = 0$. Let $\lambda > 0$. It suffices to observe that the function $f(x) = x^{\frac{\lambda-1}{p}}$ belongs to $\mathcal{L}^{p,\lambda}$ by Lemma 2.5, but the operator H_β on this function exists only when $\beta < \frac{\lambda}{p} + \frac{1}{p'}$.

With the same example $f(x) = x^{\frac{\lambda-1}{p}}$ the case of the operator $\mathcal{H}_\beta$ may be similarly considered.

$\square$

Corollary 3.2. $\|H_\beta\|_{\mathcal{L}^{p,\lambda}} \leq \frac{1}{\frac{\lambda}{p}+\frac{1}{p'}-\beta}$, $\|\mathcal{H}_\beta\|_{\mathcal{L}^{p,\lambda}} \leq \frac{1}{\beta+\frac{1-\lambda}{p}}$.

Observe that Theorem 3.1 is a particular case of the following statement for the operators of the form

$$Af(x) = \int_0^\infty a(t)f(xt)dt = \frac{1}{x} \int_0^\infty a\left(\frac{t}{x}\right) f(t)dt$$

with homogeneous kernel. Under the choice $a(t) = \chi_{[0,1]}(t)$ we obtain the operator H_β, while taking $a(t) = \chi_{[1,\infty]}(t)$ we get at the majorant of the operator $\mathcal{H}_\beta$.

Theorem 3.3. *Let* $1 \leq p < \infty$, $0 \leq \lambda < 1$ *and* $C = \int_0^\infty t^{\frac{\lambda-1}{p}} |a(t)| dt < \infty$. *Then*

$$\|Af\|_{\mathcal{L}^{p,\lambda}(\mathcal{R}_+^1)} \leq C\|f\|_{\mathcal{L}^{p,\lambda}(\mathcal{R}_+^1)}. \tag{3.3}$$

The proof follows the same lines as for Theorem 3.1.

3.2 The case of general weights

Theorem 3.4. *Let* $\varphi \in \widetilde{W} \cap (V_{++} \cup V_{-+})$. *Then the weighted Hardy operators* H_φ *and* $\mathcal{H}_\varphi$ *are bounded in the Morrey spaces* $\mathcal{L}^{p,\lambda}([0,\ell])$, $1 \leq p < \infty$, $0 \leq \lambda < 1, 0 < \ell < \infty$, *if*

$$\varphi \in \mathbb{Z}_{\frac{\lambda}{p}+\frac{1}{p'}} \quad and \quad \varphi \in \mathbb{Z}^{\frac{\lambda-1}{p}}, \tag{3.4}$$

respectively, or equivalently,

$$M(\varphi) < \frac{\lambda}{p} + \frac{1}{p'} \qquad \text{for the operator} \quad H_\varphi \tag{3.5}$$

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and

$$m(\varphi) > \frac{\lambda}{p} - \frac{1}{p} \qquad \text{for the operator} \quad \mathcal{H}_\varphi.$$

The conditions

$$m(\varphi) \le \frac{\lambda}{p} + \frac{1}{p'}, \quad M(\varphi) \ge \frac{\lambda}{p} - \frac{1}{p} \tag{3.6}$$

are necessary for the boundedness of the operators H_β and $\mathcal{H}_\beta$, respectively.

Proof. By (2.21) and (2.22), the function $\dfrac{\varphi(x)}{x^{m(\varphi)-\varepsilon}}$ is almost increasing, while $\dfrac{\varphi(x)}{x^{M(\varphi)+\varepsilon}}$ is almost decreasing for any $\varepsilon > 0$. Consequently,

$$C_1 \frac{x^{m(\varphi)-\varepsilon}}{t^{m(\varphi)-\varepsilon}} \le \frac{\varphi(x)}{\varphi(t)} \le C_2 \frac{x^{M(\varphi)+\varepsilon}}{t^{M(\varphi)+\varepsilon}}$$

and then

$$C_1 x^{m(\varphi)-\varepsilon-1} \int\limits_0^x \frac{f(t)\,dt}{t^{m(\varphi)-\varepsilon}} \le H_\varphi f(x) \le C_2 x^{M(\varphi)+\varepsilon-1} \int\limits_0^x \frac{f(t)\,dt}{t^{M(\varphi)+\varepsilon}} \tag{3.7}$$

supposing that $f(t) \ge 0$. Therefore, the operator H_φ is bounded by Theorem 3.1 for the Hardy operators with power weights, if $M(\varphi) + \varepsilon < \frac{\lambda}{p} + \frac{1}{p'}$, which is satisfied under the choice of $\varepsilon > 0$ sufficiently small, the latter being possible by (3.5). It remains to recall that condition (3.5) is equivalent to the assumption $\varphi \in \mathbb{Z}_{\frac{\lambda}{p}+\frac{1}{p'}}$ by Theorem 2.14. The necessity of the condition $m(\varphi) \le \frac{\lambda}{p} + \frac{1}{p'}$ follows from the left-hand side inequality in (3.7).

Similarly one may treat the case of the operator $\mathcal{H}_\varphi$. □

4 Weighted boundedness of the Hilbert transform in Morrey spaces

We start with the Cauchy singular integral along the real line or an interval ($\Gamma = \mathbb{R}^1$ or $\Gamma = [0, \ell]$) and denote

$$Sf(x) = \frac{1}{\pi} \int_{\mathbb{R}^1} \frac{f(t)\,dt}{t - x}, \quad x \in \mathbb{R}^1; \quad \mathbb{H}f(x) = \frac{1}{\pi} \int\limits_0^\ell \frac{f(t)\,dt}{t - x}, \quad 0 < x < \ell \le \infty. \tag{4.1}$$

In [37] there was proved the boundedness of a class of Calderon-Zygmund operators, which includes in particular the following statement.

Theorem 4.1. *The operator S is bounded in the space $\mathcal{L}^{p,\lambda}(\mathbb{R}^1)$, $1 < p < \infty, 0 \le \lambda < 1$.*

Corollary 4.2. *The Hilbert transform operator $\mathbb{H}$ is bounded in the space $\mathcal{L}^{p,\lambda}([0, \ell])$, $1 < p < \infty, 0 \le \lambda < 1$.*

4.1 Reduction of the Hilbert transform operator with weight to the Hardy operators

The boundedness of the singular operator $\mathbb{H}$ in the space $\mathcal{L}^{p,\lambda}([0,\ell],\varrho)$ with a weight ϱ is the same as the boundedness of the operator $\varrho\mathbb{H}\frac{1}{\varrho}$ in the space $\mathcal{L}^{p,\lambda}([0,\ell])$. In view of Corollary 4.2, the latter boundedness will follow from the boundedness of the operator

$$Kf(x) := \left(\varrho\mathbb{H}\frac{1}{\varrho} - \mathbb{H}\right)f(x) = \int_0^\ell K(x,t)f(t)\,dt, \tag{4.2}$$

where

$$K(x,t) := \frac{\varrho(x) - \varrho(t)}{\varrho(t)(t-x)} = \frac{\varphi(|x-x_0|) - \varphi(|t-x_0|)}{\varphi(|t-x_0|)(t-x)}$$

in the case $\varrho(x) = \varphi(|x-x_0|)$, $x_0 \in [0,\ell]$.

4.1.1 The case $x_0 = 0$

We start with the case $x_0 = 0$, so that $K(x,t) = \frac{\varphi(x)-\varphi(t)}{\varphi(t)(t-x)}$.

Lemma 4.3. *The kernel $K(x,t)$ admits the estimate*

$$|K(x,t)| \leq \begin{cases} \dfrac{C}{x}\dfrac{\varphi(x)}{\varphi(t)}, & \text{if } t < x \\ \dfrac{C}{t}, & \text{if } t > x \end{cases} \tag{4.3}$$

when $\varphi \in \mathbf{V}_{++}$, and

$$|K(x,t)| \leq \begin{cases} \dfrac{C}{x}, & \text{if } t < x \\ \dfrac{C}{t}\dfrac{\varphi(x)}{\varphi(t)}, & \text{if } t > x \end{cases} \tag{4.4}$$

when $\varphi \in \mathbf{V}_{-+}$.

Proof. Estimates (4.3)-(4.4) follow immediately from the definition of the classes $\mathbf{V}_{++}, \mathbf{V}_{-+}$. $\square$

Corollary 4.4. *The operator $K = \varrho\mathbb{H}\frac{1}{\varrho} - \mathbb{H}$ is dominated by the weighted Hardy operators*

$$|Kf(x)| \leq C\frac{\varphi(x)}{x}\int_0^x \frac{|f(t)|dt}{\varphi(t)} + C\int_x^\ell \frac{|f(t)|dt}{t} \tag{4.5}$$

when $\varphi \in \mathbb{V}_{++}$, and

$$|Kf(x)| \leq \frac{C}{x}\int_0^x |f(t)|dt + C\varphi(x)\int_x^\ell \frac{|f(t)|dt}{t\varphi(t)} \tag{4.6}$$

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when $\varphi \in \mathbb{V}_{-+}$. In particular, when $\varphi(x) = x^{\alpha}$,

$$|Kf(x)| \le \frac{C}{x} \int\limits_0^x \left(\frac{x}{t}\right)^{\max(\alpha,0)} |f(t)|dt + C \int\limits_x^{\ell} \left(\frac{x}{t}\right)^{\min(\alpha,0)} \frac{|f(t)|dt}{t}. \tag{4.7}$$

In the sequel we use the notation

$$H_{\varphi}f(x) = \frac{\varphi(x)}{x} \int\limits_0^x \frac{f(t)dt}{\varphi(t)}, \quad \mathcal{H}_{\varphi}f(x) = \varphi(x) \int\limits_x^{\ell} \frac{f(t)dt}{t\varphi(t)} \tag{4.8}$$

without fear of confusion with notation in (3.1)

Corollary 4.5. *Let $\varphi \in V_{++} \cup V_{-+}$. By (4.5)-(4.6), the boundedness of the Hardy operators H_{φ} and $\mathcal{H}_{\varphi}$ in Morrey space $\mathcal{L}^{p,\lambda}(0,\ell)$ yields that of the weighted singular operator $\varrho \mathbb{H} \frac{1}{\varrho}$, $\varrho(x) = \varphi(x)$, $1 < p < \infty$, $0 \le \lambda < 1$.*

4.1.2 The case $x_0 \ne 0$

The following simple technical fact is valid.

Lemma 4.6. *Let $-\infty \le a < b \le \infty$, $x_0 \in (a,b)$ and let $\varphi(x)$ be a non-negative function on $\mathbb{R}_+^1$. If the operator*

$$Kf(x) = \int\limits_{x_0}^b \left| \frac{\varphi(t-x_0) - \varphi(x-x_0)}{t-x} \right| \frac{f(t)}{\varphi(t-x_0)} dt, \quad x_0 < x < b$$

is bounded in the space $L^{p,\lambda}([x_0,b])$, then the operator

$$\widetilde{K}f(x) = \int\limits_a^b \left| \frac{\varphi(|t-x_0|) - \varphi(|x-x_0|)}{t-x} \right| \frac{f(t)}{\varphi(|t-x_0|)} dt, \quad a < x < b$$

is bounded in the space $L^{p,\lambda}([a,b])$.

Proof. Without loss of generality we may take $-a = b > 0$ and $x_0 = 0$. Splitting the square $Q = \{(x,t) : -a < x < a, -a < t < a\}$ into the sum of 4 squares $Q = Q_{++} + Q_{--} + Q_{-+} + Q_{+-}$, where the first sign in the index corresponds to the sign of x and the second to that of t, we reduce the boundedness of the operator $\widetilde{K}$ to that of the corresponding operators $\widetilde{K}_{++}, \widetilde{K}_{--}, \widetilde{K}_{-+}, \widetilde{K}_{+-}$. The operators $\widetilde{K}_{++}$ and $\widetilde{K}_{--}$ are bounded, the former by assumption, the latter being obviously reduced to the former. Because of the evenness of the function $\varphi(t)$, the kernels of the operators $\widetilde{K}_{-+}$ and $\widetilde{K}_{+-}$ are obviously dominated by the kernels of the operator $\widetilde{K}_{++}$, which completes the proof. $\qquad\square$

By Lemma 4.6, the validity of the statement of Corollary 4.5 in the case $x_0 \ne$ follows from the case $x_0 = 0$.

4.2 Weighted boundedness of the Hilbert transform operator; the case of power weights

Theorem 4.7. *The weighted singular operator*

$$S_\alpha f(x) = \frac{x^\alpha}{\pi} \int\limits_0^\ell \frac{f(t)\, dt}{t^\alpha (t-x)}$$

is bounded in the space $\mathcal{L}^{p,\lambda}([0,\ell])$, where $0 < \ell \leq \infty$, $1 < p < \infty$, $0 \leq \lambda < 1$, if and only if

$$-\frac{1}{p} < \alpha - \frac{\lambda}{p} < \frac{1}{p'}. \tag{4.9}$$

Proof. *The "if" part.* The case $\lambda = 0$ is well known (Babenko weighted theorem, see for instance, [19], p.30). Let $\lambda \neq 0$. By Corollary 4.2, the boundedness of S_α is equivalent to that of the difference

$$Kf(x) := (S_\alpha - S)f(x) = \frac{1}{\pi} \int\limits_0^\ell \frac{x^\alpha - t^\alpha}{t^\alpha (t-x)} f(t)\, dt.$$

By Corollary 4.5, it suffices to have the boundedness of the Hardy operators H_{β_1} with $\beta_1 = \max(\alpha, 0)$ and $\mathcal{H}_{\beta_2}$ with $\beta_2 = \min(\alpha, 0)$. Applying Theorem 3.1, we obtain that inequalities (4.9) are sufficient for the boundedness of the operator K.

The "only if" part. It suffices to consider the case $\ell < \infty$.

The necessity of condition (4.9) in the case $\lambda = 0$ is well known, see for instance [7], Lemma 4.6. Let $0 < \lambda < 1$. Suppose that $\alpha \leq \frac{\lambda-1}{p}$. To show that the operator S_α is not bounded, we choose $f(t) = t^{\frac{1-\lambda}{p}}$, which is in $\mathcal{L}^{p,\lambda}([0,\ell])$ by Lemma 2.3. Then in the case $\alpha < \frac{\lambda-1}{p}$ we have

$$S_\alpha f(x) = \frac{x^\alpha}{\pi} \int\limits_0^\ell \frac{t^{\frac{\lambda-1}{p} - \alpha}}{t-x}\, dt \sim cx^\alpha \quad \text{as} \quad x \to 0 \tag{4.10}$$

with $c = \frac{\ell^{\frac{\lambda-1}{p} - \alpha}}{\frac{\lambda-1}{p} - \alpha}$. Since $\alpha < \frac{\lambda-1}{p}$, the function $S_\alpha f(x) \sim cx^\alpha$ proves to be not in $\mathcal{L}^{p,\lambda}([0,\ell])$. In the remaining case $\alpha = \frac{\lambda-1}{p}$, the singular integral

$$\int\limits_0^\ell \frac{dt}{t-x} \sim \ln \frac{1}{x} \quad \text{as} \quad x \to 0 \tag{4.11}$$

has a logarithmic singularity and then the function $S_\alpha f(x) \sim cx^\alpha \ln \frac{2\ell}{x}$ proves to be not in $\mathcal{L}^{p,\lambda}([0,\ell])$ by Lemma 2.5.

Finally, the necessity of the condition $\alpha < \frac{\lambda}{p} + \frac{1}{p'}$ follows from the simple fact that in the case $\alpha \geq \frac{\lambda}{p} + \frac{1}{p'}$ the weighted singular integral $S_\alpha f$ does not exist on all the functions $f \in \mathcal{L}^{p,\lambda}([0,\ell])$. Indeed, take $f(t) = t^{\frac{\lambda-1}{p}} \in \mathcal{L}^{p,\lambda}([0,\ell])$, then $\frac{f(t)}{t^\alpha} = \frac{1}{t^{\alpha + \frac{1-\lambda}{p}}}$ with $\alpha + \frac{1-\lambda}{p} \geq 1$ is not in $L^1([0,\ell])$,

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while belonging of a function to L^1 is a necessary condition for the almost everywhere existence of the singular integral. □

Corollary 4.8. *Let* $-\infty \le a < b \le \infty$ *and*

$$\varrho(x) = \prod_{k=1}^{N} |x - x_k|^{\alpha_k}$$

where x_k *are arbitrary finite points in* $[a, b]$. *The singular operator* S *is bounded in the space* $L^{p,\lambda}([a, b]\varrho)$, *if and only if*

$$\frac{\lambda}{p} - \frac{1}{p} < \alpha_k < \frac{\lambda}{p} + \frac{1}{p'}, \quad k = 1, 2, ..., N. \tag{4.12}$$

Proof. The case of a singe point $x_1 = a$ when a is finite, is covered by Theorem 4.7. The case where $x_1 > a$ is treated with the help of Lemma 4.6. The reduction of the case of N points to the case of a single point is made in a standard way via a unity partition, thanks to the fact that Morrey space is a Banach function space, so that $|f(x)| \le |g(x)| \implies \|f\|_{p,\lambda} \le \|g\|_{p,\lambda}$. □

We arrive at the following result.

Theorem 4.9. *Let* $-\infty < a < b < \infty$. *The singular operator* S *is bounded in the weighted Morrey space* $\mathcal{L}^{p,\lambda}([a, b], \varrho), 1 < p < \infty, 0 \le \lambda < 1$, *with the weight*

$$\varrho(x) = \prod_{k=1}^{N} \varphi_k(|x - x_k|), \quad x_k \in [a, b]$$

where $\varphi_k \in \widetilde{W} \cap (V_{++} \cup V_{-+})$, *if*

$$\varphi_k \in \mathbb{Z}_{\frac{\lambda}{p} + \frac{1}{p'}}^{\frac{\lambda-1}{p}}, \tag{4.13}$$

or equivalently,

$$\frac{\lambda - 1}{p} < m(\varphi_k) \le M(\varphi_k) < \frac{\lambda}{p} + \frac{1}{p'}, \quad k = 1, 2, ..., N. \tag{4.14}$$

Proof. The case of a single weight $\varrho(x) = \varphi(x - a)$ follows from Theorem 3.4 by the pointwise estimates of Corollary 4.4. The case of a single weight of the form $\varrho(x) = \varphi(|x - x_0|), x_0 \in (a, b)$, is easily considered with the aid of Lemma 4.6. The passage to the case of a product of such weights is done via the standard approaches. □

Remark 4.10. When considering the case of non-power weights $\varphi(x)$, for simplicity we supposed that the interval $[0, \ell]$ for the Hardy operators or the interval $[a, b]$ for the singular operator is finite. The case of infinite interval also may be considered for non-power weights, but then we should somewhat modify definitions and introduce the Matusewska-Orlicz type indices responsible for the behavior of weights not only at the origin but also at infinity.

5 On the non-weighted boundedness of the singular Cauchy operator along Carleson curves

Our goal is to extend Theorem 4.1 to the case of the Cauchy singular operator (1.1) along Carleson curves. We will obtain such an extension from the boundedness of the maximal operator in Morrey spaces (in a more general context of metric measure spaces), making use of the pointwise estimate (1.3).

5.1 Maximal operator in Morrey spaces on metric measure spaces

The boundedness of the maximal operator

$$Mf(x) = \sup_{r>0} \int_{B(x,r)} |f(y)| d\mu(y)$$

in a certain version of Morrey spaces over metric measure spaces (X, d, μ) was proved in [5]. In the recent paper [23], the boundedness of the maximal operator on bounded metric measure space was extended to the case of variable coefficients. The boundedness of the operator M in the space $L^{p,\lambda}(X)$ under condition (2.1), may be derived from [5] and [23]. For completeness of the proof, we will present an independent and direct proof in Theorem 5.2.

The following statement well known in the Euclidean setting ([14], Lemma 1; [46], p. 53), for homogeneous spaces was proved in [39], Proposition 3.4.

Lemma 5.1. *Let X be a homogeneous metric measure space with $\mu(X) = \infty$. Then the Feffermann-Stein inequality*

$$\int_X (Mf)(y)^p \, w(y) \, d\mu(y) \leq \int_X f(y)^p \, (Mw)(y) \, d\mu(y) \tag{5.1}$$

holds for all non-negative functions f, w on X.

Theorem 5.2. *Let X be a metric measure space with $\mu(X) = \infty$. Under condition (2.1), the maximal operator M is bounded in the space $L^{p,\lambda}(X), 1 < p < \infty, 0 \leq \lambda < N$.*

Proof. We follow the main lines of the proof in [9] for the case $X = \mathbb{R}^n$. By Fefferman-Stein inequality (5.1), we obtain

$$\int_{B(x,r)} (Mf(y))^p \, dy \leq C \int_X |f(y)|^p M \chi_{B(x,r)}(y) \, dy$$

$$\leq C \int_{B(x,r)} |f(y)|^p(y) dy + C \sum_{j=0}^{\infty} \int_{B(x,2^{j+1}r) \setminus B(x,2^j r)} |f(y)|^p M \chi_{B(x,r)}(y) \, dy.$$

We make use of the estimate

$$M \chi_{B(x,r)}(y) \leq C \frac{r^N}{(d(x,y)+r)^N}, \quad x, y \in X, \tag{5.2}$$

valid under condition (2.1), which is well known in the Euclidean case and the proof in our case is in main the same as, for instance, in [8], p. 160-161, thanks to condition (2.1). We then obtain

$$\int\limits_{B(x,r)} (Mf(y))^p \, dy \le C \int\limits_{B(x,r)} |f(y)|^p(y)dy + C\sum_{j=1}^{\infty} \int\limits_{B(x,2^{j+1}r)\setminus B(x,2^j r)} |f(y)|^p M\chi_{B(x,r)}(y) \, dy$$

$$\le C \int\limits_{B(x,r)} |f(y)|^p(y)dy + \sum_{j=1}^{\infty} \frac{C}{[2^j+1]^N} \int\limits_{B(x,2^{j+1}r)} |f(y)|^p \, dy.$$

Hence

$$\|Mf\|_{\mathcal{L}^{p,\lambda}} = \sup_{x,r} \frac{1}{r^\lambda} \int\limits_{B(x,r)} (Mf(y))^p \, dy \le \sup_{x,r} \frac{C}{r^\lambda} \int\limits_{B(x,r)} |f(y)|^p dy$$

$$+ \sum_{j=1}^{\infty} \frac{C}{[2^j+1]^{N-\lambda}} \sup_{x,r} \frac{1}{r^\lambda} \int\limits_{B(x,r)} |f(y)|^p dy = C_1\|f\|_{\mathcal{L}^{p,\lambda}}. \tag{5.3}$$

$\square$

Let

$$M^\# f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}| \, d\mu(y), \quad f_{B(x,r)} = \int\limits_{B(x,r)} f(y)d\mu(y).$$

To deal with the boundedness of the singular operator via pointwise estimate (1.3), we also need the following Fefferman-Stein inequality in Morrey-norms for metric measure spaces (proved in [12] in the case $X = \mathbb{R}^n$).

Lemma 5.3. *Let X be a metric measure space with $\mu(X) = \infty$. Under condition* (2.1)

$$\|Mf\|_{L^{p,\lambda}(X)} \le C\|M^\# f\|_{L^{p,\lambda}(X)}, \quad 1 < p < \infty, 0 \le \lambda < N.$$

Proof. The proof is in fact the same as in [12]. We make use of the following weighted Fefferman-Stein inequality in L_p-norms

$$\int\limits_X |Mf(x)|^p w(x)dx \le C \int\limits_X |M^\# f(x)|^p w(x)dx, \quad w \in A_\infty, f \in L^p(X, w) \tag{5.4}$$

valid for homogeneous metric measure spaces, see [17], p. 184. According to Coifman-Rochberg [10] characterization of A_1, the function $\left[M\chi_{B(x,r)}\right]^\varepsilon$, $0 < \varepsilon < 1$, is in A_1 (see [16], Proposition 3.1 for the case of homogeneous spaces). Since $\chi_{B(x,r)} \le M\chi_{B(x,r)} \le \left[M\chi_{B(x,r)}\right]^\varepsilon$, by (5.4) we obtain

$$\int\limits_{B(x,r)} |Mf(y)|^p d\mu(y) \le C\int\limits_X |Mf(y)|^p \left[M\chi_{B(x,r)}(y)\right]^\varepsilon d\mu(y) \le C\int\limits_X |M^\# f(y)|^p \left[M\chi_{B(x,r)}(y)\right]^\varepsilon d\mu(y)$$

$$\le \int\limits_{B(x,r)} |M^\# f(y)|^p d\mu(y) + \sum_{j=0}^{\infty} \frac{C}{(2^{j+1}+1)^{N\varepsilon}} \int\limits_{B(x,2^{j+1}r)} |M^\# f(y)|^p d\mu(y),$$

where (5.2) have been used. Then similarly to estimations in (5.3) we arrive at the statement of the lemma under the choice $\varepsilon \in \left(\frac{\lambda}{n}, 1\right)$.

$\square$

5.2 Singular Cauchy operator along Carleson curves in Morrey spaces; non-weighted case

The following theorem was proved in a recent paper [23] in case of bounded curves, but in a more general setting of variable exponents.

Theorem 5.4. *Let Γ be a Carleson curve. The singular operator S_Γ is bounded in the space $L^{p,\lambda}(\Gamma)$, $\quad 1 < p < \infty, 0 \le \lambda < 1$.*

Proof. Since a function on a bounded Carleson curve may be continued by zero to an infinite Carleson curve with the preservation of the Morrey space, it suffices to consider the case where Γ is an infinite curve.

Having the pointwise estimate (1.3) in mind, we make use of the property of the norm

$$\|f\|_{p,\lambda} = \|f^s\|_{\frac{p}{s},\lambda}, \;\; 0 < s < 1$$

and have

$$\|S_\Gamma f\|_{p,\lambda} = \|(S_\Gamma f)^s\|_{\frac{p}{s},\lambda} \le \|M[(S_\Gamma f)^s]\|_{\frac{p}{s},\lambda}.$$

Then by Lemma 5.3 and estimate (1.3) we obtain

$$\|S_\Gamma f\|_{p,\lambda} \le C\|M^\#[(S_\Gamma f)^s]\|_{\frac{p}{s},\lambda} \le C\|(Mf)^s]\|_{\frac{p}{s},\lambda} = C\|Mf\|_{p,\lambda}.$$

It remains to apply Theorem 5.2. □

6 Singular Cauchy operator along Carleson curves in weighted Morrey spaces

Let Γ be a Carleson curve, $t_k \in \Gamma, k = 1, ..., N$, and ϱ a weight of form (1.2) with $\varphi_k \in \widetilde{W} \cap (V_{++} \cup V_{-+})$.

Theorem 6.1.
I) Let the curve Γ satisfy the arc-chord condition. The operator S_Γ is bounded in the Morrey space $\mathcal{L}^{p,\lambda}$, $1 < p < \infty, 0 \le \lambda < 1$, with weight (1.2), if condition (4.13) (or equivalent condition (4.14)) is satisfied.
II) Let the curve Γ satisfy the arc-chord condition and be smooth in neighborhoods of the nodes $t_k, k = 1, ..., N$, of the weight. In the case of power weights $\varphi_k(r) = r^{\alpha_k}$ the corresponding condition (4.14), that is, $\frac{\lambda-1}{p} < \alpha_k < \frac{\lambda}{p} + \frac{1}{p'}, \quad k = 1, 2, ..., N$, is also necessary for the boundedness.
III) Statement I) remains valid on a Carleson curve Γ for the space $L_*^{p,\lambda}(\Gamma)$, under the assumption that the curve Γ has the arc-chord property only at the nodes $t_k, k = 1, ..., N$ of the weight.

Proof.
I) As usual, we may consider only the case of a single weight $\varrho(t) = \varphi(|t - t_0|), t_0 \in \Gamma$. In view of Theorem 5.4, it suffices to prove the boundedness of the operator

$$Kf(t) := \left(\varrho S_\Gamma \frac{1}{\varrho} - S_\Gamma \right) f(t) = \int_\Gamma K(t, \tau) f(\tau) \, d\mu(\tau), \tag{6.1}$$

where $K(t, \tau) := \frac{\varrho(t) - \varrho(\tau)}{\varrho(\tau)(\tau - t)} = \frac{\varphi(|t - t_0|) - \varphi(|\tau - t_0|)}{\varphi(|\tau - t_0|)(\tau - t)}$. By the definition of the classes $\mathbf{V}_{++}, \mathbf{V}_{-+}$, we observe that the kernel $K(t, \tau)$ admits the estimate

$$|K(t, \tau)| \leq \begin{cases} \dfrac{C\varphi(|t - t_0|)}{|t - t_0|\varphi(|\tau - t_0|)}, & \text{if } |\tau - t_0| < |t - t_0| \\ \dfrac{C}{|t - t_0|}, & \text{if } |\tau - t_0| > |t - t_0| \end{cases} \tag{6.2}$$

when $\varphi \in \mathbf{V}_{++}$, and

$$|K(t, \tau)| \leq \begin{cases} \dfrac{C}{|t - t_0|}, & \text{if } |\tau - t_0| < |t - t_0| \\ \dfrac{C\varphi(|t - t_0|)}{|\tau - t_0|\varphi(|t - t_0|)}, & \text{if } |\tau - t_0| > |t - t_0| \end{cases} \tag{6.3}$$

when $\varphi \in \mathbf{V}_{-+}$. Then the operator K is dominated by the weighted Hardy type operators

$$|Kf(t)| \leq C\frac{\varphi(|t - t_0|)}{|t - t_0|} \int\limits_{\Gamma_t} \frac{|f(\tau)|d\mu(\tau)}{\varphi(|\tau - t_0|)} + C \int\limits_{\Gamma \backslash \Gamma_t} \frac{|f(\tau)|d\mu(\tau)}{|\tau - t_0|} \tag{6.4}$$

when $\varphi \in \mathbb{V}_{++}$, and

$$|Kf(t)| \leq \frac{C}{|t - t_0|} \int\limits_{\Gamma_t} |f(\tau)|d\mu(\tau) + C\varphi(|t - t_0|) \int\limits_{\Gamma \backslash \Gamma_t} \frac{|f(\tau)|d\mu(\tau)}{|\tau - t_0|\varphi(|\tau - t_0|)} \tag{6.5}$$

when $\varphi \in \mathbb{V}_{-+}$, where $\Gamma_t = \{\tau \in \Gamma : |\tau - t_0| < |t - t_0|\}$.

Note that the condition $\varphi \in W_1 \cap \widetilde{W}$ guarantees the equivalence

$$C_1\varphi(|s - s_0|) \leq \varphi(|t - t_0|) \leq C_2\varphi(|s - s_0|), \quad t = t(s), \ t_0 = t(s_0) \tag{6.6}$$

on curves satisfying the arc-chord condition at the point t_0. Since $\frac{\lambda}{p} + \frac{1}{p'} < 1$, condition (4.14) implies that $\varphi \in W_1$ and therefore, equivalence (6.6) holds under the conditions of the theorem.

Without a loss of generality we may assume that the the arc length counts from the point t_0, that is, $s_0 = 0$ (which may always be supposed in the case of a closed curve, while for an open curve this means that t_0 must be an end-point; the case where t_0 is not, may be easily covered similarly to Lemma 4.6). Then, in view of (6.6), it is easily seen that estimates (6.4) and (6.5) are equivalent to the following "arc-length" form

$$|Kf(t)| \leq C\frac{\varphi(s)}{s} \int\limits_0^s \frac{|f_*(\sigma)|d\sigma}{\varphi(\sigma)} + C \int\limits_s^\ell \frac{|f_*(\sigma)|}{\sigma}d\sigma, \quad t = t(s), \tag{6.7}$$

when $\varphi \in \mathbb{V}_{++}$, and

$$|Kf(t)| \leq \frac{C}{s} \int\limits_0^s |f_*(\sigma)|d\sigma + C\varphi(s) \int\limits_s^\ell \frac{|f_*(\sigma)|d\sigma}{\sigma\varphi(\sigma)}, \quad t = t(s), \tag{6.8}$$

when $\varphi \in \mathbb{V}_{-+}$ (taking into account that $s_0 = 0$), where $f_*(\sigma) = f[t(s)]$. It remains to apply Theorem 3.4 to the Hardy operators on the right-hand side of (6.7)-(6.8) keeping Remark 2.2 in mind.

II) The proof of the necessity of conditions $\frac{\lambda-1}{p} < \alpha_k < \frac{\lambda}{p} + \frac{1}{p'}$, $k = 1, 2, ..., N$, in the case of power weights follows the same line as in the proof of *the "only if"* part of Theorem 4.7, with corresponding modifications. We explain the necessary modification for (4.10). Now we have

$$S_\alpha f(t) = \frac{|t - t_0|^\alpha}{\pi} \int_\Gamma \frac{|\tau - t_0|^{-\alpha} f(\tau)}{\tau - t} d\tau$$

$$= \frac{|t - t_0|^\alpha}{\pi} \int_0^\ell \frac{|t(\sigma) - t(s_0)|^{-\alpha} f(\tau) \tau'(\sigma)}{t(\sigma) - t(s)} d\sigma, \quad t_0 = t(s_0) \in \Gamma.$$

We choose

$$f(\tau) = f[t(\sigma)] = (\sigma - s_0)_+^{\frac{\lambda-1}{p} - \alpha} \cdot \frac{t(\sigma) - t(s_0)}{\sigma - s_0} \cdot \frac{|t(\sigma) - t(s_0)|^\alpha}{t'(\sigma)}$$

where $(\sigma - s_0)_+^{\frac{\lambda-1}{p} - \alpha} = \begin{cases} (\sigma - s_0)^{\frac{\lambda-1}{p} - \alpha} &, \quad \sigma > s_0 \\ 0 &, \quad \sigma < s_0 \end{cases}$ and it is assumed that $s_0 \neq \ell$, the arguments being easily modified for the case $s_0 = \ell$. By the smoothness of the curve near the point t_0, that is, the continuity of $t'(\sigma)$ near $\sigma = s_0$ and the condition $|t'(\sigma)| \equiv 1$, we see that

$$|f(\tau)| \leq C|\tau - t_0|^{\frac{\lambda-1}{p}} \in \mathcal{L}^{p,\lambda}(\Gamma)$$

by Lemma 2.3. However, under this choice of $f(\tau)$, by the continuity of $t'(\sigma)$ it is easy to see that

$$S_\alpha f(t) \sim c|t - t_0|^\alpha \quad \text{with} \quad c \neq 0 \quad \text{as} \quad t \to t_0,$$

as in (4.10).

Finally, it remains to observe that property (4.11) of the singular integral is known to be valid on an arbitrary Carleson curve, see for instance [7], pages 118-120.

$\square$

Remark 6.2. In case one uses weights of the form $\prod_{k=1}^N \varphi_k(|s - s_k|)$, $0 \leq s_1 < s_2 < \cdots < s_N < \ell$, the requirement for Γ to satisfy the arc-chord condition in Part *III)* of Theorem 6.1 may be omitted as is easily seen from the proof.

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